The smallest degree sum that yields potentially $K_{r+1} - Z$-graphical Sequences *

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Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. A sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19, r + 1 \geq k \geq 5$, $j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices. We also determine the values of $\sigma(K_{r+1} - Z_4, n)$, $\sigma(K_{r+1} - (K_4 - e), n)$, $\sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16, r \geq 4$.

Key words: subgraph; degree sequence; potentially $K_{r+1} - Z$-graphic; potentially $K_{r+1} - Z_4$-graphic sequence
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1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d_1, d_2, ..., d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is

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called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d_1 + d_2 + ... + d_n$, and $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G+H) = V(G) \bigcup V(H)$ and $E(G+H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. We use the symbol $G[v_1, v_2, ..., v_k]$ to denote the subgraph of G induced by vertex set $\{v_1, v_2, ..., v_k\}$. We use the symbol $\epsilon(G)$ to denote the numbers of edges in graph G.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and in general by Turán [19]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5],[14],[15],[16] and [17].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2)+2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. They also pointed out that it would be nice to see where in the range for $3n-2$ to $4n-4$, the value $\sigma(K_4 - e, n)$ lies. Luo [18] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Lai [7] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin,Li and Mao[21] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r+1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li [20] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$ (In fact, Yin and Li[20] also determining the values $\sigma(K_{r+1} - ke, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$). After reading[20], using Yin-Li method Yin [22] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r+5, r \geq 3$. Lai [8] determined $\sigma(K_5 - K_3, n)$, for $n \geq 5$. Lai [9] gave a lower bound of $\sigma(K_{t+p} - K_p, n)$. Lai [10,11] determined $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for

$n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading[20] and [22], using Yin-Li method Lai and Hu[12] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r+10, r \geq 3, r+1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r+8, r \geq 3$. Using Yin-Li method Lai and Sun[13] determined $\sigma(K_{r+1} - (kP_2 \bigcup tK_2), n)$ for $n \geq 4r+10, r+1 \geq 3k+2t, k+t \geq 2, k \geq 1, t \geq 0$. To now, the problem of determining $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved. In this paper, using Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r \geq 4$ and $n \geq 5r + 16$, then

$$\sigma(K_{r+1} - K_4, n) = \sigma(K_{r+1} - (K_4 - e), n) =$$

$$\sigma(K_{r+1} - Z_4, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Theorem 1.2. If $n \geq 5r + 19, r+1 \geq k \geq 5$, and $j \geq 5$, then

$$\sigma(K_{r+1} - Z, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices.

There are a number of graphs on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices.

2 Preparations

In order to prove our main result,we need the following notations and results.

Let $\pi = (d_1, \cdots, d_n) \in NS_n, 1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{ if } d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k=(d'_1,d'_2,\cdots,d'_{n-1})$,where $d'_1\geq d'_2\geq\cdots\geq d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k. Then π'_k is called the residual sequence obtained by laying off d_k from π.

Theorem 2.1[20] Let $n\geq r+1$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r+1}\geq r$. If $d_i\geq 2r-i$ for $i=1,2,\cdots,r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2[20] Let $n\geq 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r+1}\geq r$. If $d_{2r+2}\geq r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3[20] Let $n\geq r+1$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r+1}\geq r-1$. If $d_i\geq 2r-i$ for $i=1,2,\cdots,r-1$, then π is potentially $K_{r+1}-e$-graphic.

Theorem 2.4[20] Let $n\geq 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r-1}\geq r$. If $d_{2r+2}\geq r-1$, then π is potentially $K_{r+1}-e$ -graphic.

Theorem 2.5[6] Let $\pi=(d_1,\cdots,d_n)\in NS_n$ and $1\leq k\leq n$. Then $\pi\in GS_n$ if and only if $\pi'_k\in GS_{n-1}$.

Theorem 2.6[3] Let $\pi=(d_1,\cdots,d_n)\in NS_n$ with even $\sigma(\pi)$. Then $\pi\in GS_n$ if and only if for any t,$1\leq t\leq n-1$,

$$\sum_{i=1}^{t}d_i\leq t(t-1)+\sum_{j=t+1}^{n}min\{t,d_j\}.$$

Theorem 2.7[5] If $\pi=(d_1,d_2,\cdots,d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.8[9] If $n\geq p+t$, then $\sigma(K_{p+t}-K_p,n)\geq 2[((p+2t-3)n+p+2t+1-pt-t^2)/2]$.

Lemma 2.1 [22] If $\pi=(d_1,d_2,\cdots,d_n)\in NS_n$ is potentially $K_{r+1}-e$-graphic, then there is a realization G of π containing $K_{r+1}-e$ with the $r+1$ vertices $v_1,\cdots,v_{r+1}$ such that $d_G(v_i)=d_i$ for $i=1,2,\cdots,r+1$ and $e=v_rv_{r+1}$.

Lemma 2.2 [12] Let $n\geq 2r+2$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r-2}\geq r$. If $d_{2r+2}\geq r-1$, then π is potentially $K_{r+1}-P_2$-graphic.

Lemma 2.3 Let $\pi=(d_1,\cdots,d_n)\in GS_n$ and G be a realization of π. If $\epsilon(G[v_1,v_2,...,v_{r+1}])\leq\epsilon(K_{r+1})-1$, then there is a realization H of π such that $d_H(v_i)=d_i$ for $i=1,2,\cdots,r+1$ and $v_rv_{r+1}\notin E(H)$.

The proof is similar to the proof of Lemma 2.1.

3 Proof of Main results.

Lemma 3.1. Let $n\geq 2r$ and $\pi=(d_1,d_2,\cdots,d_n)\in GS_n$ with $d_{r-1}\geq r$, $d_{r+1}\geq r-1$. If $d_i\geq 2r-i$ for $i=1,2,\cdots,r-2$, then π is potentially $K_{r+1}-e$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r$.

If $d_{r-1} \geq r+1$.

Then π is potentially $K_{r+1}-e$-graphic by Theorem 2.3.

If $d_{r-1} = r$, then $d_{r-1} = d_r = d_{r+1} = r$

Suppose π is not potentially $K_{r+1}-e$-graphic. Let H be a realization of π, then $\epsilon(H[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 2$. Let $S = (d_1, d_2, \cdots, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, \cdots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1}-graphic (Denote $S' = (d'_1, d'_2, \cdots, d'_n)$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with $v_1, v_2, \cdots, v_{r+1}$ $(d(v_i) = d_i, i = 1, 2, \cdots, r-1,$ $d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1)$, the $r+1$ vertices of highest degree containing a K_{r+1}. Hence, $G - v_{r+1}v_r$ is a realization of π. Thus, π is potentially $K_{r+1} - e$-graphic, which is a contradiction.

Case 2: $d_{r+1} = r-1$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: $d'_1 \geq 2(r-1)-1, \cdots,$ $d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1)-(r-2), d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - e$-graphic by $\{d_1 - 1, \cdots, d_{r-1} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.2. Let $n \geq 2r$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-2} \geq r+1$, $d_{r+1} \geq r$,$d_r - 1 \geq d_{d_{r+1}+2}$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r-3$, then π is potentially A_{r+1}-graphic.

Proof. The residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1, \cdots,$ $d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1)-(r-2)$, $d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially A_{r+1}-graphic by $\{d_1-1, \cdots, d_r-1\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.3 Let $n \geq 2r+2, r \geq 4$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-2} \geq r-1$ and $d_{r+1} \geq r-2$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-3$, then π is potentially $K_{r+1} - Z_4$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r-1$.

Subcase 1.1: $d_{r-1} \geq r+1$.

If $d_{r-2} \geq r+2$, then π is potentially $K_{r+1} - e$-graphic by Theorem 2.3. Hence,π is potentially $K_{r+1} - Z_4$-graphic.

If $d_{r-2} = r+1$, then $d_{r-3} - 1 \geq d_{r-2}$. The residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$, $d'_{(r-1)+1} = d'_r \geq (r-1)-1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - Z_4$-graphic by $\{d_1 - 1, \cdots, d_{r-3} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Subcase 1.2: $d_{r-1} \leq r$. then $d_{r-3} - 1 \geq d_{r-1}$. The residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq 2(r-1)-(r-3)$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$, $d'_{(r-1)+1} = d'_r \geq (r-1)-1$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - Z_4$-graphic by $\{d_1 - 1, \cdots, d_{r-3} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Case 2: $d_{r+1} = r-2$.

If $d_{r-1} < d_{r-2}$.

If $d_{r-2} \geq r$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-2$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 = d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1)-2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$, and $d'_{(r-1)+1} = d'_r = d_r \geq r-2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - Z_4$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

If $d_{r-2} = r-1$, then $d_{r-1} = d_r = r-2$ and

$$\begin{aligned}
\sigma(\pi) &\leq (r-3)(n-1) + r - 1 + (r-2)(n-r+2) \\
&= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+2) \\
&= (r-1)(2n-r) - 3(n-r) - 2
\end{aligned}$$

Hence, $\pi = ((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+2})$ and $n-r$ is even. Clearly, π is potentially $K_{r+1} - Z_4$-graphic.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} \geq d_r$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1)-2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$ and $d'_{(r-1)+1} = d'_r \geq r-2$. By Lemma 3.1, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - Z_4$-graphic by $\{d_{r-1}, d_r, d_1 - 1 \cdots, d_{r-2} - 1\} = \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

If $d_{r-1} = d_{r-2}$ and $d_{r-3} = d_r$, then $d_{r-3} = d_{r-2} = d_{r-1} = d_r \geq r+3$. Let H be a realization of π. Since $d_{r+1} = r-2$, then there is $i, j \leq r$ such that $v_{r+1}v_i, v_{r+1}v_j \notin E(H)$. Let $S = (d_1, d_2, \cdots, d_i + 1, \cdots, d_j + 1, \cdots, d_r, d_{r+1} + 2, \cdots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1}-graphic (Denote $S' = (d'_1, d'_2, \cdots, d'_n)$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with $v_1, v_2, \cdots, v_{r+1}$ ($d(v_t) = d_t, t \neq i, j, r+1$, $d(v_i) = d_i + 1, d(v_j) = d_j + 1, d(v_{r+1}) = d_{r+1} + 2$), the $r+1$ vertices of highest degree containing a K_{r+1}. Hence, $G -$

$\{v_{r+1}v_i, v_{r+1}v_j\}$ is a realization of π. Thus, π is potentially $K_{r+1} - Z_4$-graphic.

Lemma 3.4 Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-t} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - K_{1,t}$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-1} \geq r$. Then π is potentially $K_{r+1} - e$-graphic by Theorem 2.4. Hence, π is potentially $K_{r+1} - K_{1,t}$-graphic.

Case 2: $d_{r-1} \leq r-1$, that is, $d_{r-1} = r-1$, then $d_{r-1} = d_r = d_{r+1} = \cdots = d_{2r+2} = r-1$ and π'_{r+1} satisfies: $d'_{(r-1)+1} = d'_r \geq r-1$ and $d'_{2(r-1)+2} = d'_{2r} \geq (r-1)-1$. By Theorem 2.2, π'_{r+1} is potentially A_r-graphic. Therefore, π is potentially $K_{r+1} - K_{1,t}$-graphic by $\{d_1 - 1, \cdots, d_{r-t} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.5 Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with $d_{r-4} \geq r$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-2} \geq r$. Then π is potentially $K_{r+1} - P_2$-graphic by Lemma 2.2. Hence, π is potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic.

Case 2: $d_{r-2} = r-1$.

Subcase 2.1: $d_{r-3} \geq r$, then $d_{r-3} \geq d_r + 1 = d_{r+1} + 1 = r > r-1 = d_{r-2} = d_{r-1}$. Suppose π is not potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic. Let H be a realization of π, then $\epsilon(H[v_1, v_2, ..., v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \cdots, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, \cdots, d_n)$, then by Theorem 2.4, S is potentially $K_{r+1} - e$-graphic (Denote $S' = (d'_1, d'_2, \cdots, d'_n)$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ is a rearrangement of the n terms of S. Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.4). Therefore, there is a realization G of S with $v_1, v_2, \cdots, v_{r+1}$ ($d(v_i) = d_i, i = 1, 2, \cdots, r-1$, $d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1$), the $r+1$ vertices of highest degree containing a $K_{r+1} - e$ and $e = v_{r-1}v_{r-2}$ by Lemma 2.1. Hence, $G - v_{r+1}v_r$ is a realization of π. Thus, π is potentially $K_{r+1} - (P_2 \bigcup K_2)$-graphic, which is a contradiction.

Subcase 2.2: $d_{r-3} = r-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-4)(n-1) + (r-1)(n-r+4) \\ &= (r-1)(n-1) - 3(n-1) + (r-1)(n-r+1) + 3(r-1) \\ &= (r-1)(2n-r) - 3(n-r) \end{aligned}$$

Since,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r)-3(n-r)-1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)-2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Hence, π is one of the following: $((n-1)^{r-5}, (n-2)^1, (r-1)^{n-r+4})$, $((n-1)^{r-4}, (r-1)^{n-r+3}, (r-2)^1)$, for $n-r$ is odd, π is one of the following: $((n-1)^{r-4}, (r-1)^{n-r+4})$, $((n-1)^{r-6}, (n-2)^2, (r-1)^{n-r+4})$, $((n-1)^{r-5}, (n-3)^1, (r-1)^{n-r+4})$, $((n-1)^{r-5}, (n-2)^1, (r-1)^{n-r+3}, (r-2)^1)$, $((n-1)^{r-4}, (r-1)^{n-r+3}, (r-3)^1)$, $((n-1)^{r-4}, (r-1)^{n-r+2}, (r-2)^2)$, for $n-r$ is even. Clearly, π is potentially $K_{r+1}-(P_2\bigcup K_2)$-graphic.

Lemma 3.6. If $r \geq 4$ and $n \geq r+1$, then

$$\sigma(K_{r+1}-Z_4, n) \geq \sigma(K_{r+1}-K_4, n).$$

and

$$\sigma(K_{r+1}-K_4, n) \geq \begin{cases} (r-1)(2n-r)-3(n-r)+1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)+2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Proof. Obviously, for $r \geq 4$ and $n \geq r+1$, $\sigma(K_{r+1}-Z_4, n) \geq \sigma(K_{r+1}-K_4, n)$. By Theorem 2.8, for $r \geq 4$ and $n \geq r+1$, $\sigma(K_{r+1}-K_4, n) = \sigma(K_{4+(r-3)}-K_4, n) \geq 2[((4+2(r-3)-3)n+4+2(r-3)+1-4(r-3)-(r-3)^2)/2]$. Hence,

$$\sigma(K_{r+1}-K_4, n) \geq \begin{cases} (r-1)(2n-r)-3(n-r)+1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)+2, \\ \text{if } n-r \text{ is even} \end{cases}$$

Lemma 3.7. If $n \geq r+1, r+1 \geq k \geq 4$, then

$$\sigma(K_{r+1}-H, n) \geq \begin{cases} (r-1)(2n-r)-3(n-r)-1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r)-3(n-r)-2, \\ \text{if } n-r \text{ is even} \end{cases}$$

where H is a graph on k vertices which not contains a cycle on 4 vertices.

Proof. Let

$$G = \begin{cases} K_{r-3}+(\frac{n-r+1}{2}+1)K_2, \\ \text{ if } n-r \text{ is odd} \\ K_{r-3}+(\frac{n-r+2}{2}K_2\bigcup K_1), \\ \text{if } n-r \text{ is even} \end{cases}$$

Then G is a unique realization of

$$\pi = \begin{cases} ((n-1)^{r-3}, (r-2)^{n-r+3}), \\ \text{ if } n-r \text{ is odd} \\ ((n-1)^{r-3}, (r-2)^{n-r+2}, (r-3)^1), \\ \text{if } n-r \text{ is even} \end{cases}$$

and G clearly does not contain $K_{r+1} - H$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1} - H, n) \geq \sigma(\pi) + 2$. Therefore,

$$\sigma(K_{r+1} - H, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

The Proof of Theorem 1.1 According to Lemma 3.6 and $\sigma(K_{r+1} - K_4, n) \leq \sigma(K_{r+1} - (K_4 - e), n) \leq \sigma(K_{r+1} - Z_4, n)$, it is enough to verify that for $n \geq 5r + 16$,

$$\sigma(K_{r+1} - Z_4, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 16$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - Z_4$-graphic.

If $d_{r-3} \leq r - 1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-4)(n-1) + (r-1)(n-r+4) \\ &= (r-1)(n-1) - 3(n-1) + (r-1)(n-r+4) \\ &= (r-1)(2n-r) - 3(n-r) \\ &< (r-1)(2n-r) - 3(n-r) + 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-3} \geq r$.

If $d_{r-2} \leq r - 2$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-2)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ &= (r-1)(2n-r) - 3(n-r) - 3 \\ &< (r-1)(2n-r) - 3(n-r) + 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)r + \sum_{i=r+1}^{n} min\{r, d_i\} + \sum_{i=r+1}^{n} d_i \\ & = & (r-1)r + 2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)r + 2(n-r)(r-3) \\ & = & (r-1)(2n-r) - 4(n-r) \\ & < & (r-1)(2n-r) - 3(n-r) + 1, \end{array}$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - Z_4$-graphic by Lemma 3.3 or Lemma 3.4. If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-3$ such that $d_i \leq 2r-i-1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ & & +(r-2)(n+1-2r-2) \\ & = & i^2 + i(n-4r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1). \end{array}$$

Since $n \geq 5r+16$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i, attains its maximum value when $i = r-3$. Therefore,

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-3)^2 + (n-4r-2)(r-3) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1) \\ & = & (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\ & < & \sigma(\pi), \end{array}$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - Z_4, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) + 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) + 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r+16$.

The Proof of Theorem 1.2 According to Lemma 3.7, it is enough to verify that for $n \geq 5r+19$,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r+19$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - Z$-graphic.

If $d_{r-4} \leq r-1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-5)(n-1) + (r-1)(n-r+5) \\ & = & (r-1)(n-1) - 4(n-1) + (r-1)(n-r+5) \\ & = & (r-1)(2n-r) - 4(n-r) \\ & < & (r-1)(2n-r) - 3(n-r) - 2, \end{array}$$

which is a contradiction. Thus, $d_{r-4} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-3)(n-1) + (r-2)(n-r+3) \\ & = & (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ & = & (r-1)(2n-r) - 3(n-r) - 3 \\ & < & (r-1)(2n-r) - 3(n-r) - 2, \end{array}$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r} d_i + \sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)r + \sum_{i=r+1}^{n} min\{r, d_i\} + \sum_{i=r+1}^{n} d_i \\ & = & (r-1)r + 2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)r + 2(n-r)(r-3) \\ & = & (r-1)(2n-r) - 4(n-r) \\ & < & (r-1)(2n-r) - 3(n-r) - 2, \end{array}$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - Z$-graphic by Lemma 3.3 or Lemma 3.5 . If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-3$ such that $d_i \leq 2r-i-1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ & & +(r-2)(n+1-2r-2) \\ & = & i^2 + i(n-4r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1). \end{array}$$

Since $n \geq 5r+19$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a

function of i, attains its maximum value when $i = r - 3$. Therefore,

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-3)^2 + (n-4r-2)(r-3) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1) \\ & = & (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\ & < & \sigma(\pi), \end{array}$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - Z, n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{ if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, \\ \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r + 19$.

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